Exam Calculus 2

10 April 2017, 14:00-17:00

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**4+8+3=15 Points**] Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$(x, y) \mapsto \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} .$$

   (a) Show that f is not continuous at $(x, y) = (0, 0)$. (Hint: consider f along the curve parametrized as $(x(t), y(t)) = (t, t^2)$.)

   (b) Show that the directional derivatives of f exists in all directions $\boldsymbol{u} = (v, w) \in \mathbb{R}^2$, $v^2 + w^2 = 1$. Note that you will have to distinguish between the two cases $w = 0$ and $w \neq 0$.

   (c) Is f differentiable at $(x, y) = (0, 0)$? Justify your answer.

2. [**6+4+5=15 Points**] Consider the curve C parametrized by $\mathbf{r} : [0, 1] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{3}(1 + t)^{3/2}\,\mathbf{j} + \frac{1}{3}(1 - t)^{3/2}\,\mathbf{k}.$$

   (a) Determine the parametrization of C by arclength s.

   (b) At each point on the curve C, determine the curvature κ of C.

   (c) Show that the derivative of the unit tangent vector $\mathbf{T}$ with respect to arclength s is perpendicular to $\mathbf{T}$. Let $\mathbf{N}$ denote the corresponding normalized unit vector obtained from this derivative. Show that $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T} = \kappa\,\mathbf{N}$.

3. [**7+8=15 Points**] Let S be the surface in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 3$.

   (a) Show that the tangent plane of S at the point (x_0, y_0, z_0) satisfies the equation $x_0 x + y_0 y + z_0 z = 3$.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(x, y, z) = x + y - z$. Determine the maxima and minima of f on S.

4. [**5+3+5+2=15 Points**] For the constant $a \in \mathbb{R}$, consider the vector field $\mathbf{F}$ on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (axy - z^3)\,\mathbf{i} + (a - 2)x^2\,\mathbf{j} + (1 - a)xz^2\,\mathbf{k}.$$

   (a) Let $A = (0, 0, 0)$ and $B = (1, 1, 1)$, and C be the straight line segment connecting A to B. Compute the line integral of $\mathbf{F}$ along C, i.e.

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

   (b) Determine a in such a way that the vector field $\mathbf{F}$ is conservative.

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(c) Determine for the value of a found in part (b) a potential function of $\mathbf{F}$.

(d) Show that the potential function in part (c) can be used to compute the value of the line integral in part (a) in the case where $\mathbf{F}$ is conservative by using the Fundamental Theorem of Line Integrals.

5. [**10+5=15 Points**]

(a) Let S be the paraboloid $z = 9 - x^2 - y^2$ defined over the disk of radius 3 in the (x, y)-plane (i.e. the points on S have $z \geq 0$) and suppose that S is oriented by the upward pointing normal. Let $\mathbf{G}$ be the vector field on $\mathbb{R}^3$ defined by $\mathbf{G} = \nabla \times \mathbf{F}$ where

$$\mathbf{F}(x, y, z) = (2z - y)\,\mathbf{i} + (x + z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k}.$$

Compute the flux of $\mathbf{G}$ through S by using Stokes' Theorem.

(b) Show that Stokes' Theorem implies Green's Theorem.

6. [**12+3=15 Points**] Let the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$\mathbf{F}(x, y, z) = \frac{1}{3}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}).$$

(a) Let B_a be the three-dimensional ball of radius $a > 0$ centred at the origin, i.e. $B_a = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq a^2\}$. Verify Gauss' Divergence Theorem for the flux of $\mathbf{F}$ through the boundary of B_a.

(b) Show that the flux of $\mathbf{F}$ through the boundary of any solid region D in $\mathbb{R}^3$ equals the volume of D.

Solutions

1. (a) For $t \neq 0$, we have

$$f(x(t), y(t)) = \frac{x^2(t)y(t)}{x^4(t) + y^2(t)} = \frac{t^2 \cdot t^2}{t^4 + t^4} = \frac{1}{2t^2}.$$

By definition $f(0,0) = 0$. As any neighbourhood U of $(0,0)$ includes points $(x,y) \in U$ with $(x,y) \neq (0,0)$ and $y = x^2$ for which $f(x,y) = 1/2$, the function f cannot be continuous at $(x,y) = (0,0)$.

(b) By definition the directional derivative of f in the direction of $\boldsymbol{u} = (v,w)$ is

$$D_{\boldsymbol{u}} f(0,0) = \lim_{h \to 0} \frac{f(hv, hw) - f(0,0)}{h}.$$

From the definition of f it follows that for $h \neq 0$,

$$\frac{f(hv,hw) - f(0,0)}{h} = \frac{\frac{h^2v^2hw}{h^4v^4+h^2w^2} - 0}{h} = \frac{\frac{h^3v^2w}{h^4v^4+h^2w^2}}{h} = \frac{h^2v^2w}{h^4v^4 + h^2w^2} = \frac{v^2w}{h^2v^4 + w^2}.$$

which for $w \neq 0$ converges to $\frac{v^2w}{w^2} = \frac{v^2}{w}$ as $h \to 0$. For $w = 0$ and $h \neq 0$, we have

$$\frac{f(hv,hw) - f(0,0)}{h} = \frac{\frac{h^2v^2hw}{h^4v^4+h^2w^2}}{h} = 0$$

which also has a limit as $h \to 0$. The directional derivatives hence exist in all directions $\boldsymbol{u}$.

(c) The function f is not differentiable at $(x,y) = (0,0)$ as f is not continuous at $(0,0)$.

2. (a) The tangent vector associated with the parametrization is

$$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{2}(1+t)^{1/2}\,\mathbf{j} - \frac{1}{2}(1-t)^{1/2}\,\mathbf{k}$$

and has length

$$\|\mathbf{r}'(t)\| = \left(1 + \frac{1}{4}(1+t) + \frac{1}{4}(1-t)\right)^{1/2} = \left(1 + \frac{1}{2}\right)^{1/2} = \left(\frac{3}{2}\right)^{1/2}.$$

Note that we here used that $1 - t \geq 0$ for $t \in [0,1]$. For $t \in [0,1]$, the arc length is hence

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\,\mathrm{d}\tau = t\left(\frac{3}{2}\right)^{1/2}.$$

In particular $s(1) = (3/2)^{1/2}$. Inverting for t gives

$$t(s) = s\left(\frac{2}{3}\right)^{1/2}$$

with $s \in [0, (3/2)^{1/2}]$. The parametrization by arc length is hence given by

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) = s\left(\frac{2}{3}\right)^{1/2}\mathbf{i} + \frac{1}{3}\left(1 + s\left(\frac{2}{3}\right)^{1/2}\right)^{3/2}\mathbf{j} + \frac{1}{3}\left(1 - s\left(\frac{2}{3}\right)^{1/2}\right)^{3/2}\mathbf{k}.$$

(b) The unit tangent vector at $\mathbf{r}(t)$ is given by

$$\mathbf{T} = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t) = \left(\frac{2}{3}\right)^{1/2}\left(\mathbf{i} + \frac{1}{2}(1+t)^{1/2}\,\mathbf{j} - \frac{1}{2}(1-t)^{1/2}\,\mathbf{k}\right).$$

The curvature κ at $\mathbf{r}(t)$ is given by

$$\begin{aligned}
\kappa &= \frac{1}{\|\mathbf{r}'(t)\|}\left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right\| \\
&= \left(\frac{2}{3}\right)^{1/2}\left\|\left(\frac{2}{3}\right)^{1/2}\left(\frac{1}{4}(1+t)^{-1/2}\,\mathbf{j} + \frac{1}{4}(1-t)^{-1/2}\,\mathbf{k}\right)\right\| \\
&= \frac{2}{3}\left\|\left(\frac{1}{4}(1+t)^{-1/2}\,\mathbf{j} + \frac{1}{4}(1-t)^{-1/2}\,\mathbf{k}\right)\right\| \\
&= \frac{2}{3}\frac{1}{4}\left(\frac{1}{1+t} + \frac{1}{1-t}\right)^{1/2} \\
&= \frac{1}{6}\left(\frac{2}{1-t^2}\right)^{1/2}.
\end{aligned}$$

(c) As $\mathbf{T}$ has unit length it satisfies $\mathbf{T}\cdot\mathbf{T} = 1$. Differentiating both sides of this equality with respect to t gives

$$\mathbf{T}'(t)\cdot\mathbf{T}(t) + \mathbf{T}(t)\cdot\mathbf{T}'(t) = 2\mathbf{T}'(t)\cdot\mathbf{T}(t) = 0,$$

where we used the differentiation rule for the scalar product and the symmetry of the scalar product. So $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. Let $\mathbf{N}$ be the unit vector in direction of $\mathbf{T}'(t)$. Then By definition

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\|\frac{\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}}{\left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\|} = \kappa\mathbf{N}.$$

3. (a) Let $g(x,y,z) = x^2 + y^2 + z^2 - 3$. The gradient of f in $\mathbf{r}_0 = (x_0, y_0, z_0)$ is then perpendicular to the tangent plane V of S at this point. The points on V thus statisfy

$$0 = \nabla g(\mathbf{a})\cdot(\mathbf{r} - \mathbf{r}_0) = 2x_0\,(x - x_0) + 2y_0\,(y - y_0) + 2z_0\,(z - z_0).$$

The equation $xx_0 + yy_0 + zz_0 = 3$ follows from $x_0^2 + y_0^2 + z_0^2 = 3$.

(b) By the method of Lagrange multipliers there exists for a critical point $\mathbf{r}$ of f restricted to S a scalar $\lambda \in \mathbb{R}$ $\mathbf{r}$ such that

$$\begin{cases} \nabla f(\mathbf{r}) & = \lambda\,\nabla g(\mathbf{r}), \\ g(\mathbf{r}) & = 0, \end{cases}$$

or equivalently

$$\begin{cases} 1 & = 2\,\lambda x, \\ 1 & = 2\,\lambda y, \\ -1 & = 2\,\lambda z, \\ 3 & = x^2 + y^2 + z^2. \end{cases}$$

These equations cannot be satisfied for $\lambda = 0$. We can hence assume $\lambda \neq 0$. Dividing the first three equations by λ shows $x = y = -z$. From the fourth equation it then follows that $x^2 = y^2 + z^2 = 1$, i.e. $x = y = -z = \pm 1$.

As S is compact it follows from Weierstraß' Theorem that f attains its maxima and minima on S. As $f(1, 1, -1) = 3$ and $f(-1, -1, 1) = -3$ it follows that f restricted S has a maximum at $(x, y, z) = (1, 1, -1)$ and a minimum at $(x, y, z) = (-1, -1, 1)$.

4. (a) The line segment C from A to B has the parametrization $\mathbf{r}(t) = (t, t, t)$ with $t \in [0, 1]$. The line integral is then given by

$$\begin{aligned}
\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t &= \int_0^1 ((at^2 - t^3)\,\mathbf{i} + (a-2)t^2\,\mathbf{j} + (1-a)t^3\,\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\, \mathrm{d}t \\
&= \int_0^1 (at^2 - t^3 + (a-2)t^2 + (1-a)t^3)\, \mathrm{d}t \\
&= \int_0^1 ((2a-2)t^2 - at^3)\, \mathrm{d}t \\
&= \left[\frac{2a-2}{3}t^3 - \frac{a}{4}t^4\right]_{t=0}^{t=1} \\
&= \frac{2a-2}{3} - \frac{a}{4}.
\end{aligned}$$

(b) For $\mathbf{F}$ to be conservative the curl of $\mathbf{F}$ has to vanish. We have

$$\begin{aligned}
\nabla \times \mathbf{F}(x, y, z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} \\
&= (\partial_y (1-a)xz^2 - \partial_z (a-2)x^2)\,\mathbf{i} + \\
&\quad \big(\partial_z (axy - z^3) - \partial_x (1-a)xz^2\big)\,\mathbf{j} + \\
&\quad \big(\partial_x (a-2)x^2 - \partial_y (axy - z^3)\big)\,\mathbf{k} \\
&= 0\,\mathbf{i} + \big(-3z^2 - (1-a)z^2\big)\,\mathbf{j} + \big(2(a-2)x - ax\big)\,\mathbf{k}.
\end{aligned}$$

Equating this to zero gives $a = 4$.

(c) Let f denote the potential function. Then f satisfies the equations

$$f_x = 4xy - z^3, \tag{1}$$
$$f_y = 2x^2, \tag{2}$$
$$f_z = -3xz^2. \tag{3}$$

Integrating Eq. (1) with respect to x gives

$$f(x, y, z) = 2x^2y - xz^3 + g(y, z),$$

where $g(y, z)$ is a integration constant which can dependent on y and z. Differentiating with respect to y and using Eq. (2) yields

$$2x^2 + g_y(y, z) = 2x^2,$$

i.e., $g_y(y,z) = 0$. So g does not dependent on y and is hence of the form $g(y,z) = h(z)$ for some function $h : \mathbb{R} \to \mathbb{R}$. So $f(x,y,z) = 2x^2y - xz^3 + h(z)$. Differentiating with respect to z and using Eq. (3) yields

$$-3xz^2 + h'(z) = -3xz^2$$

which gives $h'(z) = 0$, i.e. h is constant. So the potential function is

$$f(x,y,z) = 2x^2y - xz^3 + c$$

with $c \in \mathbb{R}$.

(d) According to the Fundamental Theorem for Line Integrals the line integral is given by $f(B) - f(A) = f(1,1,1) - f(0,0,0) = (2-1) - 0 = 1$, where f is the potential function computed in part (c). This agrees with the result found in part (a) for $a = 4$.

5. (a) By Stokes' Theorem the flux of $\mathbf{G}$ through S is

$$\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s}. \tag{4}$$

The surface S is the part of a downward open paraboloid having $z \geq 0$. Its boundary ∂S is given the circle $x^2 + y^2 = 3$, $z = 0$. This circle has the parametrization

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 0\,\mathbf{k}$$

with $t \in [0, 2\pi]$. As S is oriented by the upward normal the parametrization $\mathbf{r}$ gives an orientation of ∂S which is consistent with that of S. The right hand side of Eq. (4) is

$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (-3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + (9\cos t - 3\sin t)\,\mathbf{k}) \cdot (-3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 0\,\mathbf{k})\,\mathrm{d}t \\
&= \int_0^{2\pi} \int_0^{2\pi} (9\sin^2 t + 9\cos^2 t)\mathrm{d}t \\
&= \int_0^{2\pi} \int_0^{2\pi} 9\mathrm{d}t \\
&= 18\pi.
\end{aligned}$$

(b) Consider a surface S contained in the (x,y)-plane oriented by the normal vector $\mathbf{k}$ with boundary ∂S. Consider a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{F}(x,y,z) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$. The curl of such a vector field is

$$\begin{aligned}
\nabla \times \mathbf{F}(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} \\
&= \big(\partial_y 0 - \partial_z P(x,y)\big)\,\mathbf{i} + \\
&\quad \big(\partial_z Q(x,y) - \partial_x 0\big)\,\mathbf{j} + \\
&\quad \big(\partial_x Q(x,y) - \partial_y P(x,y)\big)\,\mathbf{k} \\
&= 0\,\mathbf{i} + 0\,\mathbf{j} + \big(Q_x(x,y) - P_y(x,y)\big)\,\mathbf{k}.
\end{aligned}$$

The left hand side of the equality in Stokes' Theorem (4) is hence

$$\iint_S \nabla\times\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_S \big(Q_x(x,y)-P_y(x,y)\big)\,\mathbf{k}\cdot\mathbf{k}\mathrm{d}S = \iint_S \big(Q_x(x,y)-P_y(x,y)\big)\mathrm{d}A. \tag{5}$$

Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, $t \in [a,b]$ be a parametrization of ∂S where the orientation following from the tangent vector $\mathbf{r}'(t)$ is consistent with the orientation of ∂S induced by the orientation of S. This means that if we traverse ∂S by increasing t the surface S will be on our left. Due to the special form of the vector field $\mathbf{F}$ the right hand side of the equality in Stokes' Theorem (4) reduces to

$$\begin{aligned}
\int_{\partial S} \mathbf{F}\cdot\mathrm{d}\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_a^b \big(P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}\big)\cdot\big(x(t)\,\mathbf{i} + y(t)\,\mathbf{j}\big)\,\mathrm{d}t \\
&= \int_a^b \big(P(x,y)x'(t) + Q(x,y)y'(t)\big)\,\mathrm{d}t \\
&= \int_{\partial S} P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y.
\end{aligned} \tag{6}$$

Equating the right hand sides of (5) and (6) gives the equality stated in Green's Theorem.

6. (a) Gauss' Divergence Theorem states that for a solid region D in $\mathbb{R}^3$ the flux of a vector field $\mathbf{F}$ in $\mathbb{R}^3$ through the boundary ∂D of D equals the volume integral of the divergence of $\mathbf{F}$ over D, i.e.

$$\iint_{\partial D} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_D \mathrm{d}V.$$

The boundary of the ball B_a is the sphere S_a of radius a. An outward pointing unit normal vector of S_a at the point (x,y,z) is given by

$$\mathbf{n}(x,y,z) = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}).$$

The flux integral is

$$\begin{aligned}
\iint_{S_a} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \frac{1}{a}\iint_{S_a} \frac{1}{3}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\cdot\frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\,\mathrm{d}S \\
&= \frac{1}{3a}\iint_{S_a} (x^2+y^2+z^2)\,\mathrm{d}S \\
&= \frac{1}{3a}\iint_{S_a} a^2\,\mathrm{d}S \\
&= \frac{4\pi}{3}a^3,
\end{aligned}$$

where we used that the surface area of a sphere of radius a is $4\pi a^2$.

(b) The divergence of $\mathbf{F}$ at (x,y,z) is

$$\nabla\cdot\mathbf{F}(x,y,z) = \frac{1}{3}(\partial_x x + \partial_y y + \partial_z z) = 1.$$

The integration of $\nabla \cdot \mathbf{F}$ over the ball of radius a gives the volume of the ball of radius a which again is $4\pi a^3/3$.

(c) As the divergence of $\mathbf{F}$ is the function constant 1 the flux through the boundary ∂D of any solid region D in $\mathbb{R}^3$ is by Gauss' Divergence Theorem equal to the volume of the region D.